Journal of Pure and Applied Algebra 83 (1992) 23–26
North-Holland

An inverse function theorem for free associative algebras of rank two

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Communicated by J. Rhodes
Received 15 February 1992
Revised 25 March 1992

Abstract

Shpilrain, V., An inverse function theorem for free associative algebras of rank two, Journal of Pure and Applied Algebra 83 (1992) 23–26.

Let $A = K\langle x_1, x_2\rangle$ be a free associative algebra of rank 2 over an arbitrary field K. In this note, we present a 'matrix test' for an endomorphism of the algebra A to be an automorphism in the spirit of the 'inverse function theorem'. With any endomorphism ϕ of the algebra A, we associate a matrix J_ϕ ('the Jacobian matrix') with the entries from A and another matrix J_ϕ^* such that ϕ is an automorphism of A if and only if the product $J_\phi^* J_\phi$ is a nonzero scalar matrix. Invertibility of J_ϕ alone is not sufficient as a simple example shows.

Let $A = K\langle x_1, x_2\rangle$ be a free associative algebra of rank 2 over an arbitrary field K. In this note, we consider the automorphisms of A. Those have been described in [2]: the group Aut A is generated by the automorphisms of the form: (1) $x_1 \to x_2$, $x_2 \to x_1$; (2) $x_1 \to \alpha x_1$, $x_2 \to x_2$, $\alpha \in K$, $\alpha \neq 0$; (3) $x_1 \to x_1 + f(x_2)$, $x_2 \to x_2$, where $f(x_2)$ is an arbitrary polynomial.

For recognizing automorphisms among arbitrary endomorphisms of the algebra A, there is a rather convenient 'commutator test' elaborated in [2] and [3]: an endomorphism ϕ of A is an automorphism if and only if one has $[\phi(x_1), \phi(x_2)] = \alpha[x_1, x_2]$ for some nonzero element α of the field K. By $[u, v]$ we mean $uv - vu$.

We are going to present here another test for an endomorphism in the spirit of the 'inverse function theorem'. This idea is due to Birman who obtained a corresponding result for a free group of an arbitrary finite rank (see [1]). This was later generalized by Krasnikov [6]. Quite recently, similar results for Lie algebras

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have been obtained in [7] (for free Lie algebras) and in [8] for a more general situation.

With any endomorphism ϕ of the algebra A, we associate a matrix J_ϕ ('the Jacobian matrix') with the entries from A and another matrix J_ϕ^* such that ϕ is an automorphism of A if and only if the product $J_\phi^* J_\phi$ is a nonzero scalar matrix. Invertibility of J_ϕ alone is not sufficient as a simple example shows.

In order to define the matrix J_ϕ, we have to introduce some more notation. There is an augmentation homomorphism $\varepsilon : A \to A$ defined by $\varepsilon(x_i) = 0$, $i = 1,2$. The kernel of this homomorphism, which is called the augmentation ideal of A, we denote by Δ. Then there are mappings $d_i : A \to A$, $i = 1,2$ satisfying the following conditions whenever $\alpha, \beta \in K$, $u,v \in A$:

(1) $d_i(x_j) = \delta_{ij}$,

(2) $d_i(\alpha u + \beta v) + \alpha d_i(u) + \beta d_i(v)$,

(3) $d_i(uv) = d_i(u)v + u^\varepsilon d_i(v)$.

We will call these mappings *Fox derivations* in honor of R.H. Fox who gave a detailed exposition of similar mappings and their properties in the case of free group algebras (see [5]). These derivations have another nature as well. The ideal Δ is a free right A-module with a free basis $\{x_1, x_2\}$, and the mappings d_i are projections to the corresponding free cyclic direct summands. Thus any element u of Δ can be uniquely written in the form $u = x_1 d_1(u) + x_2 d_2(u)$. That is the reason why we call $d_i(u)$ a *right* Fox derivative of an element u.

One can define left Fox derivations D_i, $i = 1,2$, in a similar way using the fact that Δ is a free left A-module as well, so $u = D_1(u)x_1 + D_2(u)x_2$. These left derivations satisfy the same conditions (1) and (2) as the right ones and a slight variant of condition (3):

(3′) $D_i(uv) = D_i(u)v^\varepsilon + uD_i(v)$.

We now define the Jacobian matrix J_ϕ of an endomorphism ϕ which takes x_j to y_j as $\|D_j(y_i)\|_{1 \le i,j \le 2}$ and associated matrix J_ϕ^* as

$$\begin{Vmatrix} d_2(y_2) & -d_2(y_1) \\ -d_1(y_2) & d_1(y_1) \end{Vmatrix}.$$

Theorem. *An endomorphism ϕ of a free associative algebra A of rank 2 is an automorphism if and only if for the matrices J_ϕ and J_ϕ^* defined above, the product $J_\phi^* J_\phi$ is a nonzero scalar matrix.*

Proof. (1) Suppose ϕ is an automorphism of A. Then by the result of [2], one has $[\phi(x_1), \phi(x_2)] = \alpha[x_1, x_2]$ for some nonzero element α of the field K. Set $y_i = \phi(x_i)$. Clearly we may assume that $y_1, y_2 \in \Delta$. Take right derivatives d_2 and d_1 of both sides of this equality:

$$d_2(y_1)y_2 - d_2(y_2)y_1 = -\alpha x_1 , \tag{1}$$

$$d_1(y_1)y_2 - d_1(y_2)y_1 = \alpha x_2 . \tag{2}$$

This can be written in the matrix form as $J_\phi^* Y = \alpha IX$, where X and Y are columns: $X = (x_1; x_2)$; $Y = (y_1; y_2)$, and I is the identity matrix. Furthermore, there is a matrix identity $Y = J_\phi X$ which is a consequence of the definition of the derivatives D_i. This yields $J_\phi^* J_\phi X = \alpha IX$, and $J_\phi^* J_\phi = \alpha I$, which is a scalar matrix.

(2) Suppose now that for some endomorphism ϕ of A one has $J_\phi^* J_\phi = \alpha I$ for some nonzero element α of the field K. Multiplying both sides of this equality by the matrix X on the right, we arrive at $J_\phi^* Y = \alpha IX$. This is equivalent to the system of equalities (1) and (2). Again we may assume that $y_1, y_2 \in \Delta$. Multiplying now both sides of (1) by x_2, both sides of (2) by x_1 on the left and taking the sum of the two equalities obtained, we get $[y_1, y_2] = \alpha[x_1, x_2]$ which yields that α is an automorphism of the algebra A by the result of [3]. The proof is completed. $\square$

Remark 1. Consider the endomorphism ϕ of the algebra A defined by $\phi(x_1) = x_1 + x_2x_1x_2$, $\phi(x_2) = x_2$. Then both matrices J_ϕ and J_ϕ^* are invertible whence ϕ is clearly not an automorphism of A. Thus the automorphisms of a free associative algebra of rank 2 are determined by matrix 'quasi identity' $J_\phi^* J_\phi = \alpha I$, and α here plays the role of the determinant of the noncommutative Jacobian matrix J_ϕ. It can be checked in a straightforward way that the composition of two automorphisms takes this parameter to the product of corresponding parameters of these automorphisms, but it also becomes clear immediately upon recalling that this α comes from the equality $[\phi(x_1), \phi(x_2)] = \alpha[x_1, x_2]$.

Remark 2. It is possible to associate a single matrix with any endomorphism ϕ of the algebra A, such that it is invertible if and only if ϕ is an automorphism of A [4], but this matrix has the entries not from A, but from some enveloping algebra which is the A-bimodule $A \otimes_K A$ with multiplication defined by $(a \otimes b)(c \otimes d) = ac \otimes db$, and extending by linearity.

Acknowledgment

I am grateful to V. Drensky for many useful discussions.

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